ISSN 2229-5518

Numerical Investigation of Fredholm Integro-Differential Equations by STHWS Method

S. Sekar and C. Jaisankar

Abstract— In this paper presents a method based on Haar wavelet approximation using single-term Haar wavelet series (STHWS) to obtain approximate numerical solution of Fredholm integro-differential equations of first [26] and second order [5]. The numerical results obtained by the present method compares favorably with those obtained by various methods earlier in the literature. The obtained discrete solutions were compared with exact solutions of the Fredholm integro-differential equations of first [26] and second order [5] and Runge-Kutta method (RK) to highlight the efficiency of the STHWS.

Index Terms— Integro-differential equations, Fredholm Integro-differential equations, Haar wavelets, Runge-Kutta method, Single-term Haar wavelet series.

1 Introduction

Integro-Differential Equation (IDE) is an important branch of modern mathematics and arises frequently in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory and electrostatics (Kurt and Sezer, 2008). IDE is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function. It can be classified into Fredholm equations and Volterra equations. The upper bound of the region for integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type. In this study, we focus on first and second order linear Fredholm integrodifferential equation.

In the engineering field, numerical approaches were practiced to obtain an approximation solution for the problem [1]. To solve a linear integro-differential equation numerically, discretization of integral equation to the solution of system of linear algebraic equations is the basic concept used by researchers to solve integro-differential problems. By considering numerical techniques, there are many methods can be used to discretize problem [1] such as compact finite difference (Zhao and Corless, 2006), Wavelet-Galerkin (Avudainayagam and Vani, 2000), variational iteration method (Sweilam, 2007) rationalized Haar functions (Maleknejad *et al*., 2004), Tau, (Hosseini and Shahmorad, 2003), Lagrange interpolation (Rashed, 2003), piecewise approximate solution (Hosseini and Shahmorad, 2005), conjugate gradient (Khosla and Rubin, 1981), quadraturedifference (Fedotov, 2009), variational (Saad and Schultz, 1986), collocation (Aguilar and Brunner, 1988), homotopy perturbation (Yildirim, 2008) and Euler-Chebyshev method (Van der Houwen and Sommeijer, 1997). Earlier numerical treatment has been done for first order integro-differential equation (Aruchunan and Sulaiman, 2009).

In this conjunction, there are many iterative methods under the category of Krylov subspaces have been proposed widely to be one of the feasible and successful classes of numerical algorithms for solving linear systems. Actually, there are several Krylov subspaces iterative methods can be considered such as Conjugate Gradient (CG) (Hestenes and Stiefel, 1952), Generalized Minimal Residual (GMRES) (Saad and Schultz, 1986), Conjugate Gradient Squared (Sonneveld, 1989), Bi-Conjugate Gradient Stabilized (Bi-CGSTAB) (Van der Vorst, 1992) and Orthogonal Minimization (ORTHOMIN) (Vinsome, 1976).

STHWS plays an important role in both the analysis and numerical solution of differential inclusions. STHWS can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. However, working with integro-differential equations places special demands on STHWS codes. In recent years, there has been an increased interest in several methods were arisen to solve the Fredholm integro-differential equations. STHWS can have a significant impact on what is considered a practical approach and on the types of problems that can be solved. S. Sekar and team of his researchers [17 - 23] introduced the STHWS to study the time-varying nonlinear singular systems, analysis of the differential equations of the sphere, to study on CNN based hole-filter template design, analysis of the singular and stiff delay systems and nonlinear singular systems from fluid dynamics, numerical investigation of nonlinear volterra-hammerstein integral equations, to study on periodic and oscillatory problems, and numerical solution of nonlinear problems in the calculus of variations.

Wavelets theory is a relatively new and an emergine area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly Wavelets are very successfully used in signal analysis for Wave form representa-

————————————————

- *Department of Mathematics, Government Arts College (Autonomous), Cherry road, Salem – 636 007, Tamil Nadu, India E-mail:sekar_nitt@rediffmail.com*
- *Department of Mathematics, A.V.C. College (Autonomous), Mannampandal, Mayiladuthurai – 609 305, Tamil Nadu, India*

tions and segmentations, time-frequency analysis and fast algorithm for easy implementation. Han Danfu introduced CASWavelet method to solve integro-differential equations [7]. P. Darania introduced a method for solve integro-differential equations [4].

In this work we apply STHWS method to solve linear Fredholm integro-differential equations of first [26] and second order [] and we will show that convergent rate of STHWS is more accelerate than the methods presented in [26,].

2 Preliminaries

Consider the linear Fredholm integro-differential equation as

$$y'(x) = f(x) + \int_a^b k(x,t)y(t)dt$$

with initial conditions

$$y(0) = y_0$$

and the second order linear Fredholm integro-differential equation as

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) + \int_a^b k(x,t)y(t)dt = f(x)$$

with initial conditions

$$y'(0) = n, \quad y(0) = m$$

Where the functions

$p(x), q(x), r(x)$ = Constant matrices
$f(x)$ = A given vector function, the kernel
$k(x,t)$ = A given matrix function
$y(x)$ = The solution to be determined

3 Single-Term Haar Wavelet Series Method

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves with magnitude of ± 1 in some intervals and zeros elsewhere [17]. In general,

$$\left.\begin{aligned} &h_n(t) = h_1(2^j t - k), n = 2^j + k, \\ &j \geq 0, 0 \leq k < 2^j, n, j, k \in Z \end{aligned}\right\}$$

$$h_1(t) = \begin{cases} 1, 0 \leq t < \frac{1}{2} \\ -1, \frac{1}{2} \leq t < 1 \end{cases},$$

Namely, each Haar wavelet contains one and just one square wave, and is zero elsewhere. Just these zeros make Haar wavelets to be local and very useful in solving stiff systems. Any function *y(t)*, which is square integrable in the interval [0,1). Can be expanded in a Haar series with an infinite number of terms

$$\left.\begin{aligned} &y(t) = \sum_{i=0}^{\infty} c_i h_i(t), i = 2^j + k, \\ &j \geq 0, 0 \leq k < 2^j, n, j, t \in [0,1] \end{aligned}\right\} \quad (1)$$

where the Haar coefficients

$$c_i = 2^j \int_0^1 y(t)h_i(t)dt$$

are determined such that the following integral square error ε is minimized:

$$\left.\begin{aligned} &\varepsilon = \int_0^1 \left[y(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt, \\ &m = 2^j, j \in \{0\} \cup N \end{aligned}\right\}$$

usually, the series expansion Equation (1) contains an infinite number of terms for a smooth *y(t)*. If *y(t)* is a piecewise constant or may be approximated as a piecewise constant, then the sum in Eq. (1) will be terminated after *m* terms, that is

$$\begin{aligned} &y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = c_{(m)}^T h_{(m)}(t), t \in [0,1] \\ &c_{(m)}(t) = [c_0 c_1 \ldots c_{m-1}]^T, \\ &h_{(m)}(t) = [h_0(t) h_1(t) \ldots h_{m-1}(t)]^T, \end{aligned} \quad (2)$$

where "T" indicates transposition, the subscript *m* in the parantheses denotes their dimensions. The integration of Haar wavelets can be expandable into Haar series with Haar coefficient matrix *P[3]*.

$$\int h_{(m)}(\tau)d\tau \approx P_{(m\times m)} h_{(m)}(t), t \in [0,1]$$

where the m-square matrix *P* is called the operational matrix of integration and single-term $P_{(1\times1)} = \frac{1}{2}$. Let us define [17]

$$h_{(m)}(t)h_{(m)}^T(t) \approx M_{(m\times m)}(t),$$

and $M_{(1\times1)}(t) = h_0(t)$. Equation (3) satisfies

$$M_{(m\times m)}(t)c_{(m)} = C_{(m\times m)}h_{(m)}(t),$$

where $c_{(m)}$ is defined in Equation (2) and $C_{(1\times1)} = c_0$.

4 Numerical Examples for Fredholm Integro-Differential Equations of First Order

To show the efficiency of the STHWS, we have considered the following problem taken from [26], with step size $h = 0.1$ along with the exact solutions. The discrete solutions obtained by the two methods, STHWS and the Adomian's Decomposition Method (ADM); the absolute errors between them are tabulated and are presented in Table 1 - 3. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of

" x "and are presented in Fig. 1 to Fig. 3 for the following problems, using three dimensional effects.

Example 4.1 Cosider the following linear Fredholm integro-differential equation [26]

$$y'(x) = xe^{x} + e^{x} - x + \int_0^1 xy(t)dt$$

with initial conditions

$$y(0) = 0$$

The exact solution is as follows:

$$y(x) = xe^{x}$$

TABLE 1
EXACT AND DISCRETE SOLUTIONS OF EXAMPLE 4.1

x	Example 4.1		
	Exact Solutions	ADM Solutions	STHWS Solution
0	0	0	0
0.1	0.1105171	0.1105408	0.1105173
0.2	0.2442806	0.2443243	0.244281
0.3	0.4049576	0.4050213	0.4049582
0.4	0.5967299	0.5968136	0.5967307
0.5	0.8243606	0.8244643	0.8243616
0.6	1.0932713	1.093495	1.0932725
0.7	1.4096269	1.4097706	1.4096283
0.8	1.7804327	1.7805964	1.7804343
0.9	2.2136428	2.2138265	2.2136446
1	2.7182818	2.7184855	2.7182838

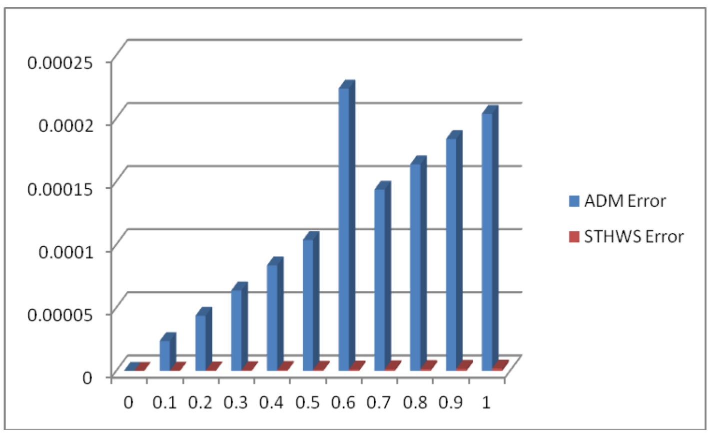

Fig. 1. Error graph for Example 4.1

Example 4.2 Cosider the following linear Fredholm integro-differential equation [26]

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t)dt$$

with initial conditions

$$y(0) = 0$$

The exact solution is as follows:

$$y(x) = x$$

TABLE 2
EXACT AND DISCRETE SOLUTIONS OF EXAMPLE 4.2

x	Example 4.2		
	Exact Solutions	ADM Solutions	STHWS Solution
0	0	0	0
0.1	0.1	0.1000237	0.1000002
0.2	0.2	0.2000437	0.2000004
0.3	0.3	0.3000637	0.3000006
0.4	0.4	0.4000837	0.4000008
0.5	0.5	0.5001037	0.500001
0.6	0.6	0.6001237	0.6000012
0.7	0.7	0.7001437	0.7000014
0.8	0.8	0.8001637	0.8000016
0.9	0.9	0.9001837	0.9000018
1	1	1.0002037	1.000002

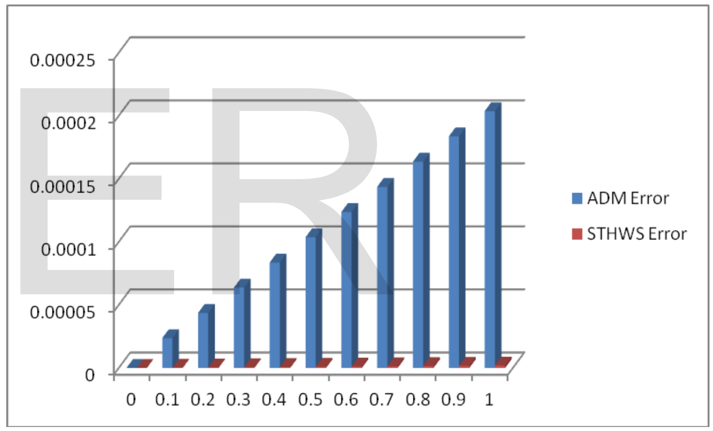

Fig. 2. Error graph for Example 4.2

Example 4.3 Cosider the following linear Fredholm integro-differential equation [26]

$$y'(x) = \int_0^1 [\sin(4\pi x + 2\pi t)y(t)dt + y(x) - \cos(2\pi x)$$
$$- 2\pi\sin(2\pi x) - (1/2)\sin(4\pi x)]$$

with initial conditions

$$y(0) = 1$$

The exact solution is as follows:

$$y(x) = \cos(2\pi x)$$

TABLE 3
EXACT AND DISCRETE SOLUTIONS OF EXAMPLE 4.3

x	Example 4.2		
	Exact Solutions	ADM Solutions	STHWS Solution

0	1	1	1
0.1	-0.1279637	-0.12794	-0.1279635
0.2	-0.9672506	-0.9672069	-0.9672502
0.3	0.3755096	0.3755733	0.3755102
0.4	0.8711474	0.8712311	0.8711482
0.5	-0.5984601	-0.5983564	-0.5984591
0.6	-0.7179851	-0.7178614	-0.7179839
0.7	0.7822121	0.7823558	0.7822135
0.8	0.5177956	0.5179593	0.5177972
0.9	-0.9147302	-0.9145465	-0.9147284
1	-0.2836911	-0.2834874	-0.2836891

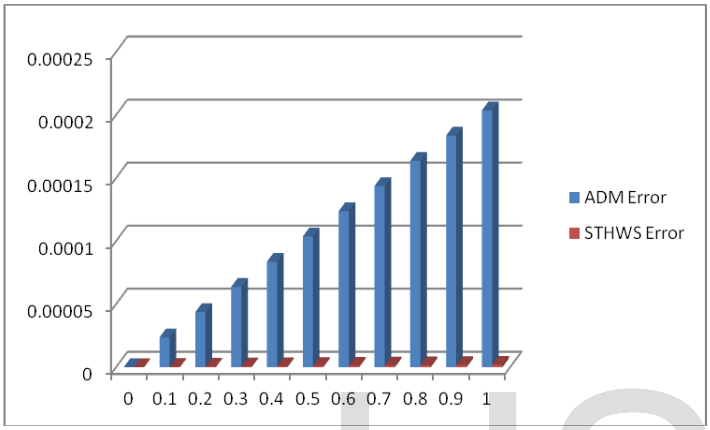

Fig. 3. Error graph for Example 4.3

5 Numerical Example for Fredholm Integro-Differential Equations of Second Order

To show the efficiency of the STHWS, we have considered the following problem taken from [5], with step size $h = 0.1$ along with the exact solutions. The discrete solutions obtained by the two methods, STHWS and the Generalized Minimal Residual (GMRES) methods; the absolute errors between them are tabulated and are presented in Table 1 - 4. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of " x "and are presented in Fig. 1 to Fig. 6 for the following problem, using three dimensional effects.

Example 5.1 Cosider the following linear Fredholm integro-differential equation [5]

$$y''(x) = 9y(x) + \frac{e^{-15} - 1}{3} + \int_0^5 y(t)dt, x \in [0,5]$$

with initial conditions

$$y(0) = 1, y'(0) = -3$$

The exact solution is as follows:

$$y(x) = e^{-3x}$$

TABLE 4
Exact and Discrete Solutions of Example 5.1

	Example 5.1		
x	Exact Solutions	GMRES Solutions	STHWS Solution
0	1	1	1
0.1	0.7408182	0.7408419	0.7408184
0.2	0.5488116	0.5488553	0.548812
0.3	0.4065697	0.4066334	0.4065703
0.4	0.3011942	0.3012779	0.301195
0.5	0.2231302	0.2232339	0.2231312
0.6	0.1652989	0.1654226	0.1653001
0.7	0.1224564	0.1227001	0.1224578
0.8	0.090718	0.0908817	0.0907196
0.9	0.0672055	0.0673892	0.0672073
1	0.0497871	0.0499908	0.0497891

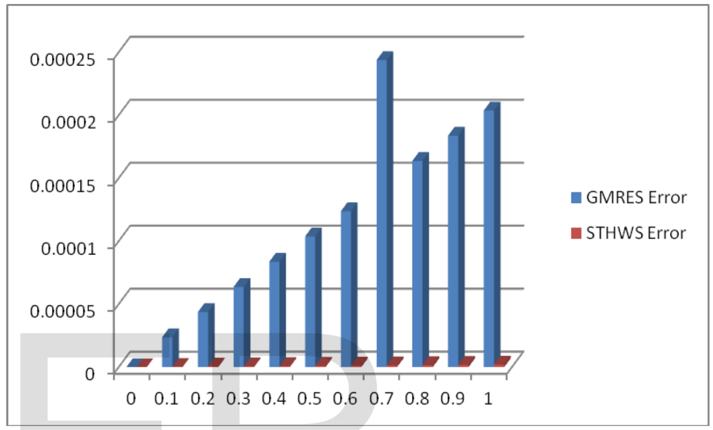

Fig. 4. Error graph for Example 5.1

6 Conclusion

In this paper, a new numerical method for solving Fredholm integro-differential equations of first and second order proposed. Here the Fredholm integro-differential equations of first and second order are solved by using STHWS method. From the numerical examples, we could conclude that the proposed method almost coincides with the exact solutions of the problems. From the Table 1-4 and Figures 1-4, compare to ADM [26] and GMRES [5] methods STHWS method gave better results and shows the STHWS method efficiency.

Acknowledgment

The authors gratefully acknowledge the Dr.A.R.Rajamani, Principal, Government Arts College (Autonomous), Salem-636 007, for encouragement and support. The authors also thank Mr.R.P.Sampathkumar, Associate Professor and Head of the Department of Mathematics, Government Arts College (Autonomous), Salem-636 007, Tamil Nadu, India, for his kind help and encouragement.

References

[1] Aguilar, M. and H. Brunner, 1988. Collocation methods for second-order Volterra integro-differential equations. Applied Numer. Math., 4: 455-470. DOI: 10.1016/0168-9274(88)90009-8

[2] Aruchunan, E. and J. Sulaiman, 2009. Numerical solution of first kind linear Fredholm Integrodifferential equation using conjugate gradi-

ent method. Proceedings of the Curtin Sarawak 1st International Symposium on Geology, Sept. 2009, pp: 11-13.

[3] Avudainayagam, A. and C. Vani, 2000. Wavelet-Galerkin method for integro-differential equations. Applied Math. Comput., 32: 247-254. DOI: 10.1016/S0168-9274(99)00026-4

[4] P. Darania, Ali Ebadian, A method for the numerical solution of the integro-differential equation, APP. Math. comput. 188 (2007), 657-668.

[5] Elayaraja Aruchunan and Jumat Sulaiman, "Numerical Solution of Second-Order Linear Fredholm Integro-Differential Equation Using Generalized Minimal Residual Method", American Journal of Applied Sciences 7 (6): 780-783, 2010

[6] Fedotov, A.I., 2009. Quadrature-difference methods for solving linear and nonlinear singular integrodifferential equations. Nonlinear Anal., 71: 303-308. DOI: 10.1016/j.na.2008.10.075

[7] Han Danfu,shang Xufeng, Numerical solution of integro-differential equations by applying CAS Wavelet operational matrix of integration, APP. Math. comput. 194 (2007), 460-466

[8] Hestenes, M. and E. Stiefel, 1952. Methods of conjugate gradients for solving linear system. J. Res. Nat. Bur. Stand., 49: 409-436. http://nvl.nist.gov/pub/nistpubs/jres/049/6/V49.N06.A08.pdf

[9] Hosseini, S.M. and S. Shahmorad, 2003. Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases. Applied Math. Model, 27: 145-154. DOI: 10.1016/S0307-904X(02)00099-9

[10] Hosseini, S.M. and S. Shahmorad, 2005. Numerical piecewise approximate solution of Fredholm integro-differential equations by the Tau method. Applied Math. Model., 29: 1005-1021. DOI:10.1016/j.apm.2005.02.003

[11] Khosla, P.K. and S.G. Rubin, 1981. A conjugate gradient iterative method. Comput. Fluids, 9: 109-121. DOI: 10.1016/0045-7930(81)90020-7

[12] Kurt, N. and M. Sezer, 2008. Polynomial solution of high-order Fredholm integro-differential equations with constant coefficients. J. Franklin Inst., 345: 839-850. DOI: 10.1016/j.jfranklin.2008.04.016

[13] Maleknejad, K., F. Mirzaee and S. Abbasbandy, 2004. Solving linear integro-differential equations system by using rationalized Haar functions method. Applied Math. Comput., 155: 317-328. DOI: 10.1016/S0096-3003(03)00778-1

[14] Rashed, M.T., 2003. Lagrange interpolation to compute the numerical solutions differential and integrodifferential equations. Applied Math. Comput., 151: 869-878. DOI: 10.1016/S0096-3003(03)00543-5

[15] Saad, Y. and M.H. Schultz, 1986. GMRES: A generalized minimal residual algorithm for solving non-symmetric linear sytems. SIAM J. Sci. Stat. Comput., 7: 856-869. DOI: 10.1137/0907058

[16] Saad, Y., 2003. Iterative Methods for Sparse Linear Systems. Society for Industrial and Applied Mathematics (SIAM). ISBN: 0898715342, pp: 158.

[17] S. Sekar and A. Manonmani, "A study on time-varying singular nonlinear systems via single-term Haar wavelet series", *International Review of Pure and Applied Mathematics*, vol.5, pp. 435-441, 2009.

[18] S. Sekar and G.Balaji, "Analysis of the differential equations on the sphere using single-term Haar wavelet series", *International Journal of Computer, Mathematical Sciences and Applications*, vol.4, pp.387-393, 2010.

[19] S. Sekar and M. Duraisamy, "A study on CNN based hole-filler template design using single-term Haar wavelet series", *International Journal of Logic Based Intelligent Systems*, vol.4, pp.17-26, 2010.

[20] S. Sekar and K. Jaganathan, "Analysis of the singular and stiff delay systems using single-term Haar wavelet series", *International Review of Pure and Applied Mathematics*, vol.6, pp. 135-142, 2010.

[21] S. Sekar and R. Kumar, "Numerical investigation of nonlinear volterra-hammerstein integral equations via single-term Haar wavelet series", *International Journal of Current Research*, vol.3, pp. 099-103, 2011.

[22] S. Sekar and E. Paramanathan, "A study on periodic and oscillatory problems using single-term Haar wavelet series", *International Journal of Current Research*, vol.2, pp. 097-105, 2011.

[23] S. Sekar and M. Vijayarakavan, "Analysis of the non-linear singular systems from fluid dynamics using single-term Haar wavelet series", *International Journal of Computer, Mathematical Sciences and Applications*, vol.4, pp. 15-29, 2010.

[24] Sonneveld, P., 1989. CGS A fast Lanczos-type solver for nonsymmetric linear systems. SIAM J. Sci.Stat.Comput., 10: 36-52. DOI: 10.1137/0910004

[25] Sweilam, N.H., 2007. Fourth order integro-differential equations using variational iteration method. Comput. Math. Appli, 54: 1086-1091. DOI: 10.1016/j.camwa.2006.12.055

[26] A. R. Vahidi, E. Babolian, Gh. Asadi Cordshooli and Z. Azimzadeh, " Numerical Solution of Fredholm Integro-Differential Equation by Adomian's Decomposition Method", Int. Journal of Math. Analysis, Vol. 3, 2009, no. 36, 1769 – 1773.

[27] Van der Houwen, P.J. and B.P. Sommeijer, 1997. Euler-Chebyshe methods for integro-differential equations. Applied Numer. Math., 24: 203-218. DOI: 10.1016/S0168-9274(97)00021-4

[28] Van der Vorst, H., 1992. Bi-CGSTAB: A fast and smoothly converging variant of Bi-CG for the solution of non-symmetric linear systems. SIAM J. Sci. Stat.Comput., 13: 631-644. DOI: 10.1137/0913035

[29] Vinsome, P.K.W., 1976. ORTHOMIN, An iterative method for solving sparse sets of simultaneous linear equations. Proceedings of SPE Symposium on Numerical Simulation of Reservoir Performance, Feb. 19-20, Los Angeles, California, pp: 5729.

[30] Yildirim, A., 2008. Solution of BVPs for fourth-order integro differential equations by using homotopy perturbation method. Comput. Math. Appli., 32:1711-1716. DOI: 10.1016/j.advwatres.2009.09.003

[31] Zhao, J. and R.M. Corless, 2006. Compact finite difference method for integro-differential equations. Applied Math. Comput., 177: 325-328. DOI: 10.1016/j.amc.2005.11.007